IJESRT

INTERNATIONAL JOURNAL OF ENGINEERING SCIENCES & RESEARCH TECHNOLOGY

Some Integral Representations of Generalized Hypergeometric Polynomial Set

V.S. Bhagavan

Department of Mathematics, K.L. University, Guntur-522502,A.P., India

drvsb002@kluniversity.in

Abstract

In this paper , the author has derived some important properties of generalized hypergeometric polynomial set namely, an integral representation of various types including addition and multiplication theorems, finite difference formula. Furthermore, some classical orthogonal polynomials like , Laguerre, Mexiner, Gottlieb, Krawtchouk and Mexiner-pollaczek polynomials are also obtained as particular cases of generalized hypergeometric polynomials. These polynomials play an important role in various branches of engineering, science and technology and also constitute good models for many systems in various fields.

Mathematics Subject Classification: 33C45, 33C50, 33C80

Keywords: Special Functions, Hypergeometric Polynomials, Integral Representations.

Introduction

Recently, I.K. Khanna and V.S.Bhagavan [3] studied some properties of $u_n(\beta;\gamma;x)$ such as generating functions with the help of the representation theory of SL(2,C) i.e., a complex special linear group. It is a worth noting that the polynomial set $u_n(\beta;\gamma;x)$ is a product of x^n and hypergeometric function which enable to derive different types of generating functions. Because of the important role which hypergeometric polynomials/functions play in problems of physics and applied mathematics , the theory of generating functions has been developed into various directions and found wide applications in various branches of analysis namely infinite series , linear differential equations, statistics distributions, operations research and functions of a complex variables . The hypergeometric functions have also retained its significance in science and technology . In this paper , an attempt is made to derive integral representations of various types to the generalized hypergeometric polynomial set $u_n(\beta;\gamma;x)$ including addition and multiplication theorems and finite difference formula.

The principle interest in our results lies in the fact that a number of special cases would yield inevitably to many new and known results of the theory of special functions of various classical orthogonal polynomials namely the Laguerre , Meixner, Gottlieb, Krawchouk and Mexiner-Pollaczek polynomials are derived as the special cases of our results

Definition

S.D.Bajpai and M.S.Arora [2] studied the semi-orthogonality property and an integral involving Fox's H-function of $u_n(\beta;\gamma;x)$ defined as

$$u_n(\beta;\gamma;x)$$

$$= x^n \; {}_2F_1[-n,\beta;\gamma;\frac{1}{x}] \,, \quad (2.1)$$

where n is a non- negative integer , x is any non-zero complex variable and β,γ are independent of n.

Remark: If β,γ are dependent of n then many properties which are valid for β,γ independent β of n fail to be valid for β,γ dependent upon n.

The aim of the present paper is to studied some more interesting classical proprieties of this function such as addition, multiplication formulae, finite difference formula and integral representations of the various types. The function $u_n(\beta;\gamma;x)$ satisfies the differential equation

$$\{x(1-x)\,D^2-[(n+\beta-1)-(\gamma+2n-2)x]-n(\gamma+n-1)\}\,u_n(x) \;=0,$$

$$where\; u_n(x)=u_n(\beta;\gamma;x) \;\; and\; D\equiv\frac{d}{dx} \qquad (2.2)$$

Applications

1. $\lim_{\beta\to\infty}\left\{\beta^{-n}u_n\left(\beta;1+\alpha;\frac{\beta}{x}\right)\right\}=\frac{n!}{(1+\alpha)_n}x^{-n}L_n^{\alpha}(x),$ (2.3)

where $L_n^{\alpha}(x)$ is the Laguerre polynomial . [9]

2. $u_n\left(-Y;\gamma;(1-\rho^{-1})^{-1}\right)=(1-\rho^{-1})^{-n}M_n(Y;\gamma,\rho)$, provided $\gamma>0, 0<\rho<1, Y=0,1,2.....$ (2.4)

where $M_n(Y;\gamma,\rho)$ is the Mexiner polynomial.[10]

3. $u_n(-Y;1;(1-e^{\lambda})^{-\lambda})=(e^{-\lambda}-1)^{-n}\phi_n(Y,\lambda),$ (2.5)

where $\phi_n(Y,\lambda)$ is the Gottlieb polynomial.[9]

4. $u_n(-Y;-N;P)=P^nK_n(Y;P,N),$ (2.6)

where $K_n(Y;P,N)$ is the Krawtchouk polynomial.[10]

5. $u_n(\lambda+iy;2\lambda;(1-e^{-2i\phi})^{-1})=\frac{n!}{(2\lambda)_n}(2i)^{-n}\operatorname{cosec}^n\phi\, P_n^{\lambda}(y;\phi),$ (2.7)

where $P_n^{\lambda}(y;\phi)$ is the Miexner –pollaczek polynomial.[10]

Preliminaries: To find the addition , multiplication formulae , finite difference formula and integral representations of the various types, we have used the well known results [8] :

1. $D^n(u,v)=\sum_{k=0}^{n}\binom{n}{k}\left(D^{n-k}u\right)\left(D^kv\right),$

2. f(x+y)= $\sum_{n=0}^{\infty}\frac{f^n(x)}{n!}y^n$

3. f(xy)= $\sum_{n=0}^{\infty}\frac{(y-1)^nx^nf^n(x)}{n!}$

where $|y|<\rho,\rho$ being the radius of convergence of the analytic function f(x).

4. $\Delta_{\alpha}\{f(\alpha)\}=f(\alpha+1)-f(\alpha),$

Addition and Multiplication Formulae

Theorem: Prove that

$$u_n(\beta;\gamma;x+y)=\sum_{k=0}^{\infty}\binom{n}{k}u_{n-k}(\beta;\gamma;x)\ y^k \quad (2.8)$$

and

$$u_n(\beta;\gamma;xy)=\sum_{k=0}^{\infty}\binom{n}{k}u_{n-k}(\beta;\gamma;x)(y-1)^kx^k. \quad (2.9)$$

Proof: We have $u_n(\beta;\gamma;x+y)=\sum_{k=0}^{\infty}\frac{d^k}{dx^k}\ \frac{1}{k!}\ [u_n(\beta;\gamma;x)]\ y^k$

$$=\sum_{k=0}^{\infty}\sum_{p=0}^{n}\frac{(-n)_p(\beta)_p(n-p)!x^{n-p-k}\ y^k}{k!\,p!(n-p-k)!(\gamma)_p}$$

$$= \sum_{k=0}^{\infty} \binom{n}{k} u_{n-k}(\beta;\gamma;x)\ y^k \text{ , which is same as (2.8).}$$

$$u_n(\beta;\ \gamma;\ xy) = \sum_{k=0}^{\infty} \frac{1}{k!}(y-1)^k\ x^k\ \frac{d^k}{dx^k} u_n(\beta;\gamma;x).$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k(-n)_k}{k!}\ u_{n-k}(\beta;\gamma;x)(y-1)^k x^k$$

$$= \sum_{k=0}^{\infty} \binom{n}{k} u_{n-k}(\beta;\gamma;x)(y-1)^k x^k.$$

Hence the proof of the theorem.

Finite Difference Formula

Theorem: Prove that

$$u_n(\beta+\lambda;\gamma+\lambda;x) = \frac{(-1)^n\Gamma(\gamma+\lambda)\,x^{n+\lambda}}{\Gamma(\beta+\lambda)}\Delta_\lambda^n\left\{\frac{\Gamma(\beta+\lambda)}{\Gamma(\gamma+\lambda)}x^{-\lambda}\right\}, \tag{2.10}$$

$$where\ \ \Delta_\lambda^n f(\lambda) = f(\lambda+1) - f(\lambda) \quad and \quad \Delta_\lambda^n f(\lambda) = \sum_{k=0}^{n}(-1)^{(n-k)}\binom{n}{k} f(\lambda+k).$$

Proof: Since $u_n(\beta;\gamma;x) = \sum_{k=0}^{n}\frac{(-n)_k(\beta)_k}{k!(\gamma)_K}x^{n-k}$

$$= \frac{(-1)^n\Gamma(\gamma)}{\Gamma(\beta)}\sum_{k=0}^{n}\frac{(-1)^{n-k}\Gamma(\beta+k)}{\Gamma(\gamma+k)}\binom{n}{k}x^{n-k}.$$

Now, writing $\beta+\lambda$ *and* $\gamma+\lambda$ *for* β *and* γ respectively, we have

$$u_n(\beta+\lambda;\gamma=\lambda;x) = \frac{(-1)^n\Gamma(\gamma+\lambda)\,x^{n+\lambda}}{\Gamma(\beta+\lambda)}\sum_{k=0}^{n}\frac{(-1)^{n-k}\Gamma(\beta+\lambda+k)}{\Gamma(\gamma+\lambda+k)}\binom{n}{k}x^{-\lambda-k}$$

$$u_n(\beta+\lambda;\gamma+\lambda;x) = \frac{(-1)^n\Gamma(\gamma+\lambda)\,x^{n+\lambda}}{\Gamma(\beta+\lambda)}\Delta_\lambda^n\left\{\frac{\Gamma(\beta+\lambda)}{\Gamma(\gamma+\lambda)}x^{-\lambda}\right\},$$

Hence the theorem.

Integral Representations

The following types of integral representations for the polynomial set $u_n(\beta;\gamma;x)$ have been discussed

I. Contour integral representation,
II. Real integral representation,
III. Infinite single integral representation
And
IV. Finite single integral representation

The existence of these representations directly depend upon the uniform convergence of the integrals.

I .Contour Integral Representation

Consider the generating relation [2] for the polynomial set $u_n(\beta;\gamma;x)$ i.e.,

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n\,\mathrm{u}_\mathrm{n}(\beta;\gamma;x)\,t^n}{n!}=(1-xt)^{-\alpha}\,{}_2F_1\left[\alpha,\beta;\gamma;-\frac{t}{1-xt}\right]. \quad (2.11)$$

Let us write

$$\mathrm{f(t)}=(1-x\,t)^{-\alpha}\,{}_2F_1\left[\alpha,\beta;\gamma;-\frac{t}{1-x\,t}\right]. \quad (2.12)$$

$$\mathrm{f(t)}=\sum_{n=0}^{\infty}\frac{f^n(0)\,t^n}{n!},$$

By using the Maclaurin's theorem

and we find that the coefficients

$$f^n(0)=\frac{n!}{2\pi i}\int^{(+0)}\frac{f(t)}{t^{n+1}}dt, \qquad n=0,1,2,3,\ldots\ldots \quad (2.13)$$

Thus from (2.11) ,(2.12) and (2.13), we arrive at the following theorem.

$$Theorem:\ If\quad (1-xt)^{-\alpha}\,{}_2F_1\left[\alpha,\beta;\gamma;-\frac{t}{1-xt}\right]=\sum_{n=0}^{\infty}\frac{(\alpha)_n u_n(\beta;\gamma;x)t^n}{n!}.$$

Then

$$u_n(\beta;\gamma;x)=\frac{n!}{2\pi i(\alpha)_n}\int^{(0+)}t^{-n-1}(1-xt)^{-\alpha}\,{}_2F_1\left[\alpha,\beta;\gamma;-\frac{t}{1-xt}\right]dt, \quad (2.14)$$

where the contour of integration encircles the origin of the t-plane in the positive direction.

II.**Real Integral Representation**

If, in equation (2.14), we replace the contour t by $e^{i\theta}$ then we get

$$u_n(\beta;\gamma;x)=\frac{n!}{2\pi(\alpha)_n}\int^{2\pi}e^{in\theta}(1-xe^{i\theta})^{-\alpha}\,{}_2F_1\left[\alpha,\beta;\gamma;-\frac{e^{i\theta}}{1-xe^{i\theta}}\right]d\theta$$

$$=\frac{n!}{2\pi(\alpha)_n}\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{k!(\gamma)_k}\int_0^{2\pi}e^{-in\theta}(1-xe^{i\theta})^{-\alpha}\left[\frac{-e^{i\theta}}{1-xe^{i\theta}}\right]^k d\theta$$

$$=\frac{n!}{2\pi(\alpha)_n}\sum_{k=0}^{\infty}\frac{(-1)_k(\alpha)_k(\beta)_k}{k!(\gamma)_k}\int_0^{2\pi}e^{(k-n)i\theta}(1-xe^{i\theta})^{-\alpha-k}d\theta$$

$$=\frac{n!}{2\pi(\alpha)_n}\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)_k(\alpha)_k(\alpha+k)_s(\beta)_k}{k!s!(\gamma)_k}\int_0^{2\pi}e^{(k-n)i\theta}(xe^{i\theta})^s d\theta$$

$$=\frac{n!}{2\pi(\alpha)_n}\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)_k(\alpha)_k(\alpha+k)_s(\beta)_k x^s}{k!s!(\gamma)_k}\int_0^{2\pi}e^{(k-n+s)i\theta}d\theta$$

$$=\frac{n!}{2\pi(\alpha)_n}\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)_k(\alpha)_{k+s}\,(\beta)_k x^s}{k!s!(\gamma)_k}\int_0^{2\pi}Cis\{k-n+s)\theta\}d\theta, \quad (2.15)$$

$$Where\ \ Cis\Phi=\cos\Phi+i\sin\Phi.$$

III. **Infinite single integral representation.**

As we know that
$$u_n(\beta;\gamma;x) = \sum_{k=0}^{n} \frac{(-n)_k(\beta)_k x^{n-k}}{k!(\gamma)_k}$$
$$= \sum_{k=0}^{n} \frac{(-n)_k \Gamma(\beta+k-\tfrac{1}{2}+\tfrac{1}{2})\, x^{n-k}}{k!\,(\gamma)_k \Gamma(\beta)}$$
$$= \sum_{k=0}^{n} \frac{(-n)_k x^{n-k}}{k!(\gamma)_k \Gamma(\beta)} \int_{-\infty}^{\infty} \exp(-t^2)\, t^{2(\beta+k-\frac{1}{2})} dt,$$

Since
$$\Gamma(\rho-k+\tfrac{1}{2}) = \int_{-\infty}^{\infty} \exp(-t^2)\, t^{2(\rho-k)} dt$$
$$= \frac{x^n}{\Gamma(\beta)} \int_{-\infty}^{\infty} \exp(-t^2)\, t^{2\beta-1} {}_1F_1\left[-n;\ \gamma;\ \frac{t^2}{x}\right] dt.$$

Thus , we have

THEOREM. If $\mathrm{Re}(\beta) > \tfrac{1}{2}$, *then*
$$u_n(\beta;\gamma;x) = \frac{x^n}{\Gamma(\beta)} \int_{-\infty}^{\infty} \exp(-t^2)\, t^{2\beta-1} {}_1F_1\left[-n;\ \gamma;\ \frac{t^2}{x}\right] dt. \tag{2.15}$$

IV. Finite Single Integral Representation.

We know that
$$u_n(\beta;\gamma;x) = \sum_{k=0}^{n} \frac{(-n)_k(\beta)_k x^{n-k}}{k!(\gamma)_k}$$
$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \sum_{k=0}^{n} \frac{(-n)_k x^{n-k}}{k!} \int_0^1 t^{\beta+k-1}(1-t)^{\gamma-\beta-1} dt$$

since
$$\frac{(a)_k}{(b)_k} = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 t^{a+k-1}(1-t)^{b-a-1} dt$$
$$= \frac{\Gamma(\gamma)x^n}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}\left(1-\frac{t}{x}\right)^n dt$$
$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(x-t)^n dt.$$

Thus, we conclude

THEOREM. *If* $\mathrm{Re}(\beta) > 0$ *and* $\mathrm{Re}(\gamma-\beta) > 0$, *then*
$$u_n(\beta;\gamma;x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(x-t)^n dt.$$

Remark: In a similar way, one can be deduced many more representations namely Finite Double Integral Representation and Infinite Double Integral Representation etc., which are of great importance in the theory of special functions of mathematical physics.

References

[1] Bride Mc,E.B. *Obtaining Generating Functions.* Springer –Verlag, New York, Heidelberg and Berlin(1971).

[2] Bajpai, S.D.and Arora M.S..*Semi-orthogonality of a class of the Gauss's hypergeometric polynomials*. Anna.Math.Blaisc Pascal 1(1),75-83, (1994).

[3] Khanna , I.K. and Srinivasa Bhagavan, V. *Generating Functions of Gegenbaur Polynomials* $C_n^{\ v}(x)$ *from a view point of Lie Algebra. Proc. Math. Soc.* 12, 93-98 (1996).

[4] Khanna , I.K. and Srinivasa Bhagavan, V. *Weisener 's method to obtain generating relations for the generalized polynomial set .J. Phys . A : Math. Gen*.32, 989-998 (1999).

[5] Khanna , I.K. and Srinivasa Bhagavan, V and Singh M. N: *Generating relations of the hypergeometric functions by Lie group-theoretic method*, J. Mathematical Physics , Analysis and Geometry, 3, 287-303(2000)

[6] Khanna , I.K. and Srinivasa Bhagavan, V: *Lie group-theoretic origins of certain generating functions of the generalized hypergeometric polynomials*, Integral Transforms and Special Functions, Vol.II , No. 2 , 177-188(2001)

[7] Mathai, A.M. *A Handbook of Generalized Special Functions for Statistics and Physical Sciences*, Clarendon Press, Oxford(1993).

[8] Miller, W. Jr. *Lie Theory and Special Functions*. Academic Press ,New York and London(1968).

[9] Rainville, E.D. *Special Functions*. Macmillan, New York; Reprinted by Chelsea Publ. Co., Bronx, New York,(1971).

[10] Srivastava, H.M. and Manocha H.L. *A Treatise on Generating Functions*. Ellis Harwood Ltd(1984).

[11] Wesiner, L. *Group-theoretic origin of certain generating functions.* Pacific J. Math. 5, 1033-1039 (1955).